9-1-1996

Exchange and Optimality

S. Ghosal

Heracles M. Polemarchakis

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1133

EXCHANGE AND OPTIMALITY

S. Ghosal and H. Polemarchakis

September 1996

Exchange and Optimality[1]

S. Ghosal[2]
and
H.M. Polemarchakis[3]

August 1996

Abstract

A feasible social state is irreducible if and only if, for any non-trivial partition of individuals with two groups, there exists another feasible social state at which every individual in the first group is equally well-off and someone strictly better-off.

Competitive equilibria decentralize irreducible pareto optimal social states.

Keywords: social states, optimality, exchange.

J.E.L. classification numbers: C70, C72, D60, D62.

[1]This text presents research results of the Belgian programme on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsability is assumed by its authors.The Commission of the European Union provided additional support through the HCM grant ERBCHRXCT-940458.

[2]Department of Economics, Queen Mary and Westfield College, University of London, Mile End Road, London E1-4NS, U.K.

[3]CORE, Université Catholique de Louvain, 34 voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium.

Exchange and Optimality

S. Ghosal and H.M. Polemarchakis

1. The theorems of classical welfare economics[4] identify competitive equilibrium allocations with pareto optimal allocations, which makes the case for the market as a mechanism for the allocation of resources.

For competitive equilibrium allocations to be pareto optimal, it is essential that individuals not be satiated and, under uncertainty, that the asset market for the transfer of revenue across date-events be complete. For pareto optimal allocations to be competitive equilibrium allocations it is essential that the economy be convex. It is essential for both that no external effects arise across individuals, which we focus on in this paper.

External effects do not arise when the objects of choice of each individual are the only arguments of his utility function, while his choice is restricted only by a budget constraint that reflects aggregate scarcity.

Altruism or envy, which allow the consumption of one individual, chosen and paid for by him, to affect positively or adversly the utility of another, or public goods, for which the consumption of one individual is augmented by the consumption of another, constitute external effects and interfere with the identification of competitive equilibria with pareto optima.

It was Lindahl[5] who first realized that optimal allocations in the presence of external effects can be identified with competitive equilibrium allocations for a particular extension of the set of commodities and markets. It suffices to consider the arguments of the utility function of each individual as objects of choice distinct from the objects of choice of other individuals and price them correctly, requiring that at equilibrium, and possibly only there, the choices of individuals concerning an object of collective choice coincide. Individualized markets can hardly be competitive,[6] the revelation of the private informations of individuals necessary for the implementa-

[4]Arrow (1951), Debreu (1951).
[5]Lindahl (1919, 1928).
[6]Arrow (1970).

tion of competitive outcomes is not evident,[7] even the convexity of the economy is problematic.[8] Nevertheless, Lindahl equilibria remain an illuminating benchmark.

Here, we remain in the framework of Lindahl. We consider an abstract set of social states as the domain of the preferences of individuals and we pose the question whether pareto optimal states can obtain as competitive equilibria of an associated economy.

Importantly, we do not require that there be any private goods. This is called for if social states are interpreted as complete descriptions of economic or social activity as is the case in social choice theory. Normative arguments often adopt a view point prior to any economic or social acitivity and, in this context, private goods would not be appropriate.

In the economy which we associate with the set of social states, using the construction of Lindahl, neither the free disposal of commodities nor the non-satiation of preferences are satisfied.

Two insights stand out:

i) For an arbitrary distribution of revenue, competitive equilibria need not exist.
ii) Though standard conditions, importantly convexity, suffice for a pareto optimal social state to be a quasi-equilibrium, the social state must be "irreducible" in order to be a competitive equilibrium.

In simple arguments,[9] the distribution of revenue does not affect the equilibrium. Here, it may prevent the existence of a competitive equilibrium.

At a quasi-equilibrium, individuals minimize expenditure for the level of utility attained, while at a competitive equilibrium they maximize utility for the expenditure incurred. In the absence of external effects, a condition of irreducibility[10] is employed to guarantee the passage from quasi-equilibria to competitive equilibria and avoid the alternative, and less appealing, assumption that the pareto optimum or the endowment are effectively interior to the consumptions sets of individuals.[11]

[7] Groves and Ledyard (1977).
[8] Starrett and Zeckhauser (1974).
[9] Coase (1960).
[10] McKenzie (1959, 1961).
[11] Arrow and Debreu (1954).

An economy, with only private goods, is irreducible if and only if "however we may partition the consumers into two groups, if the first group receives an aggregate trade which is an attainable output for the rest of the market, the second group has within its feasible aggregate trades one which if added to the goods already obtained by the first group could be used to improve the position of someone in the group while damaging the position of none there."[12] Irreducibility guarantees that if "some individual has income in the sense that he can trade with a budget whose value is below zero,"[13] which is the case by construction, all individuals do, and then, by a standard argument, a quasi-equilibrium is a competitive equilibrium. An economy with private and public goods or externalities can be defined to be irreducible if the private sector is irreducible, which suffices.

In our framework, with preferences defined over social states and without recourse to private goods, which would simplify the argument,[14] the analogue of irreducibility may not seem evident, but it is : a feasible social state is irreducible if and only if, for any partition of the individuals into two groups, there exists an alternative feasible social state at which every individual in the first group is equally well off and someone strictly better off. So stated, the definition could apply equally well to an economy with private goods: the economy is irreducible if any individually rational allocation is irreducible. Irreducible pareto optimal social states are competitive equilibria of an associated economy.

2. Individuals are $h \in \mathbf{H} = \{1, \cdots, H\}$, a finite set.

Social states are $s \in \mathbf{R}^N$, euclidean space of finite dimension.

An individual is described by the pair $(\mathbf{S}^h, u^h)$, where $\mathbf{S}^h \subset \mathbf{R}^N$ is the domain of social states for the individual, and $u^h : \mathbf{S}^h \rightarrow \mathbf{R}$ is his ordinal utility function.[15]

The set of feasible social states is $\mathbf{S} \subsetneq \mathbf{R}^N$.

A society is

$$\boldsymbol{\Sigma} = \{\mathbf{H}, (\mathbf{S}^h, u^h) : h \in \mathbf{H}, \mathbf{S}\}.$$

[12] McKenzie [1959; pp.58, 59].

[13] McKenzie [1959; p.58].

[14] Foley (1970), Mas-Colell (1980), Milleron (1972).

[15] The representation of preferences by a utility function is not an essential loss of generality.

Assumption [Compatibility]: The set of feasible social states is contained in the domain of social states of each individual.

Social states not in the domain of social states for some individual can be ignored.

At a social state, $\overline{s}$, an individual is satiated if and only if $u^h(\overline{s}) \geq u^h(s)$, for all $s \in \mathbf{S}^h$. The individual is locally satiated at $\overline{s}$ if and only if there exists a social state $\overline{\overline{s}}$ and a neighborhood of $\overline{\overline{s}}$, $\mathbf{V}^h \subset \mathbf{S}^h$, such that $u^h(\overline{\overline{s}}) \geq u^h(s)$, for all $s \in \mathbf{V}^h$; this is stronger than the standard notion of local non-satiation, since the state $\overline{\overline{s}}$ may be different from the state $\overline{s}$.

At a social state s, for a set of individuals $\hat{\mathbf{H}} \subset \mathbf{H}, u^{\hat{\mathbf{H}}}(s) = (u^h(s) : h \in \hat{\mathbf{H}})$.

The social state s pareto dominates the state s' if and only if $u^{\mathbf{H}}(s) > u^{\mathbf{H}}(s')$, and it dominates it strongly if and only if $u^{\mathbf{H}}(s) \gg u^{\mathbf{H}}(s')$.

A feasible social state is pareto optimal if and only if no feasible state pareto dominates it, and it is weakly pareto optimal if and only if no feasible state pareto dominates it strongly.

Assumption [Continuity]: The set of feasible social states is compact. The utility function of every individual is continuous.

If the assumption of continuity fails, pareto optimal social states, even weakly pareto optimal ones, need not exist.

Lemma 1: Under the assumptions of compatibility and continuity, pareto optimal social states exist.

Proof: Let $\overline{s} \in \mathbf{S}$ and let $\mathbf{S}^0 = \{s \in \mathbf{S} : u^{\mathbf{H}}(s) \geq u^{\mathbf{H}}(\overline{s})\}$, which is non-empty and compact.

Consider the nested sequence of sets $\overline{\mathbf{S}}^0 \supset \cdots \supset \overline{\mathbf{S}}^{h-1} \supset \overline{\mathbf{S}}^h \supset \cdots \supset \overline{\mathbf{S}}^H$, where $\overline{\mathbf{S}}^h = \{s \in \mathbf{S}^{h-1} : u^h(s) \geq u^h(s')$, for all $s' \in \overline{\mathbf{S}}^{h-1}\}$ is non-empty and compact.

Social states in $\overline{\mathbf{S}}^1$ are weakly pareto optimal, which thus exist.

Social states in $\overline{\mathbf{S}}^H$ are pareto optimal. If not, suppose $s^* \in \overline{\mathbf{S}}^H$ while $u^{\mathbf{H}}(\hat{s}) > u^{\mathbf{H}}(s^*)$, for some $\hat{s} \in \mathbf{S}$; in particular, $u^{\hat{h}}(\hat{s}) > u^{\hat{h}}(s^*)$. Since $u^{\hat{h}}(s^*) \geq u^{\hat{h}}(s)$, for all

$s \in \overline{\mathbf{S}}^{\hat{h}-1}$, $\hat{s} \notin \overline{\mathbf{S}}^{\hat{h}-1}$. Since $\hat{s} \notin \overline{\mathbf{S}}^{\hat{h}-1}$, while $u^{\hat{h}-1}(\hat{s}) \geq u^{\hat{h}-1}(s^*) \geq u^{\hat{h}-1}(s)$, for all $s \in \mathbf{S}^{\hat{h}-2}$, by the definition of $\overline{\mathbf{S}}^{\hat{h}-1}$, $\hat{s} \notin \overline{\mathbf{S}}^{\hat{h}-2}$. Similarly, $\hat{s} \notin \overline{\mathbf{S}}^{\hat{h}-3}, \cdots, \hat{s} \notin \overline{\mathbf{S}}^{0}$. But this is a contradiction, since $u^{\mathbf{H}}(\hat{s}) > u^{\mathbf{H}}(s^*) \geq u^{\mathbf{H}}(\overline{s})$. ■

The procedure in the proof of lemma 1 generates all pareto optimal or weakly pareto optimal social states by varying $\overline{s}$, the initial social state.

3. The economy associated with the society $\mathbf{\Sigma}$ is specified as follows:

Agents in the economy are individuals, $h \in \mathbf{H}$, and a firm.

The set of commodity bundles is $\mathbf{R}^{HN} = \Pi_{h \in \mathbf{H}} \mathbf{R}^N$.

The firm is characterized by its technology, $\mathbf{Y} \subset \mathbf{R}^{HN}$, a set of $y = (\cdots, y_h, \cdots)$, production bundles.

An individual is described by the pair $(\mathbf{Z}^h, u^h)$, where $\mathbf{Z}^h \subset \mathbf{R}^{HN}$ is the exchange set for the individual, a set of $z = (\cdots, z_h, \cdots)$, exchange bundles, and $u^h : \mathbf{Z}^h \to \mathbf{R}$ is his ordinal utility function.

The characteristics of individuals and of the firm reflect the set of social states and the preferences of individuals.

The technology of the firm is

$$\mathbf{Y} \; = \; \{y \in \mathbf{R}^{HN} : y_h \in \mathbf{S}, \text{ and } y_h = y_{h'}, h, h' \in \mathbf{H}\}.$$

The exchange set of an individual is

$$\mathbf{Z}^h \; = \; \{z \in \mathbf{R}^{HN} : z_h \in \mathbf{S}^h, \text{ while } z_{h'} = 0, h' \in \mathbf{H} \setminus \{h\}\}.$$

Thus, there is an unambiguous association of a social state with an exchange bundle, a bundle in the exchange set of the individual, and vice versa. The utility function of the individual over exchange bundles is defined by $u^h(z) = u^h(z_h)$.[16]

The economy associated with the society $\mathbf{\Sigma}$ is thus

$$\mathbf{E}(\mathbf{\Sigma}) \; = \; \{\mathbf{H}, \mathbf{Y}, (\mathbf{Z}^h, u^h) : h \in \mathbf{H}\}.$$

[16] It is pedantic to use different symbols for the utility functions on $\mathbf{Z}^h$ and $\mathbf{S}^h$.

A state of the economy is an array $a = (y, \cdots, z^h, \cdots)$ of $y \in \mathbf{Y}$, a production plan for the firm, and, for each individual, of $z^h \in \mathbf{Z}^h$, an exchange bundle.

A state of the economy is feasible if and only if $\Sigma_{h \in \mathbf{H}} z^h = y$. From the structure of the technology of the firm and the exchange sets of individuals, an allocation is feasible if and only if there is a feasible social state, s, such that $z^h_h = s$, for every individual, while $y = (\cdots, s, \cdots)$, for the firm. Thus, there is an unambiguous association of a feasible social state with a feasible state of the economy, and vice versa.

The set of commodity prices is $\mathbf{R}^{HN}$, as dual to the space of commodity bundles. Commodity prices are $p = (\cdots, p_h, \cdots)$. The value of a commodity bundle $z = (\cdots, z_h, \cdots)$ at prices $p = (\cdots, p_h, \cdots)$ is $pz = \Sigma_{h \in \mathbf{H}} p_h z_h$.

4. A competitive equilibrium is a pair, (p^*, a^*), of commodity prices and a feasible allocation, such that

i) $p^* y^* \geq p^* y$, for all $y \in \mathbf{Y}$;
ii) $u^h(z) > u^h(z^{h*}) \Rightarrow p^* z > p^* z^{h*}$, for all $h \in \mathbf{H}$.

A quasi-equilibrium is a pair, (p^*, a^*), of commodity prices and a feasible allocation, such that

i) $p^* y^* \geq p^* y$, for all $y \in \mathbf{Y}$;
ii) $u^h(z) \geq u^h(z^{h*}) \Rightarrow p^* z \geq p^* z^{h*}$, for all $h \in \mathbf{H}$;
iii) $u^h(z) > u^h(z^{h*}) \Rightarrow p^* z > p^* z^{h*}$, for some $h \in \mathbf{H}$.

At a competitive equilibrium, all individuals maximize utility, while at a quasi-equilibrium, all individuals minimize expenditure and some, simultaneously, maximize utility. At either, the firm maximizes profit.

At a competitive equilibrium, (p^*, a^*), such that the associated feasible social state, s^*, is an interior point of the set of feasible social states, $\sum_{h \in \mathbf{H}} p^*_h = 0$.

That, at a quasi-equilibrium, some individuals maximize utility is necessary to prevent the definition from being vacuous, with $p^* = 0$.

With one individual, a competitive equilibrium is a quasi-equilibrium; weak pareto optimality and pareto optimality coincide .

A feasible allocation, a^*, is a competitive equilibrium allocation if and only if (p^*, a^*) is a competitive equilibrium for some prices, p^*. Similarly, a^* is a quasi-equilibrium allocation if and only if (p^*, a^*) is a quasi-equilibrium.

Competitive equilibria or quasi-equilibria as defined here do not fix the distribution of revenue across individuals. For this reason, the ownership of commodities or shares of firms by individuals is not part of the specification of the economy.

A distribution of revenue is $\tau = (..., \tau^2, ...)$. A competitive equilibrium or a quasi-equilibrium for a distribution of revenue is such that, for every individual, $p^* z^h \leq \tau^h$, and, as a consequence,$p^* y^h * \leq \sum_{h \in H} \tau^h$.

With a fixed distribution of revenue, competitive equilibria need not exist.

Example 1: The set of feasible social states is $\mathbf{S} = \{s : -1 \leq s \leq 1\} \subset \mathbf{R}$. Individuals are $h \in \mathbf{H} = \{1, 2\}$ and have utility functions $u^1 = s, u^2 = -(s - 1/2)^2$, with domain the set of feasible social states. Suppose the distribution of revenue is fixed and the revenue of each individual is 0. At prices $p = (p_1, p_2)$, the budget constraint of an individual is $p_h z_h^h \leq 0$. If $p_1 + p_2 = 0$, the budget constraints take the form $p_1 z_1^1 \leq 0$ and $p_1 z_2^2 \geq 0$, for individuals $h = 1$ and $h = 2$, respectively. If $p_1 = 0, z_1^1 = 1$ while $z_2^2 = 1/2$, if $p_1 > 0, z_1^1 = 0$ while $z_2^2 = 1/2$, and if $p_1 < 0, z_1^1 = 1$ while $z_2^2 = 0$, all contradictions. Alternatively, suppose $p_1 + p_2 > 0$. From the profit maximization of the firm, $y = (1, 1)$. If $z_1^1 = 1$ and $z_2^2 = 1$, as equilibrium would require, it follows from the budget constraints of individuals that $p_1 \leq 0$ and $p_2 \leq 0$ or $p_1 + p_2 \leq 0$, a contradiction. Similarly, $p_1 + p_2 < 0$ leads to a contradiction. Thus, competitive equilibrium prices, where $z_1^1 = z_2^2 \in \mathbf{S}$, do not exist. ∎

Though competitive equilibria do not exist for some distribution of revenue, they do for other distributions. In particular, if τ^h is the revenue of individual h, with $\tau^1 + \tau^2 = 0$, it is a straightforward computation that distributions of revenue with $1/2 \leq \tau^h \leq 1$ support the allocations associated with social states $s \in [1/2, 1]$ at competitive equilibrium allocations.

It is a consequence of proposition 1, below, that, under standard assumptions, there exist distributions of revenue for which competitive equilibria exist.

At prices p, the bundle $\overline{z} \in \mathbf{Z}^h$ is a minimum expenditure bundle for the individual if and only if $pz \geq pz^h$ or, equivalently, $p_h z_h \geq p_h z_h^h$, for all $z \in \mathbf{Z}^h$.

At the bundle $\overline{z} \in \mathbf{Z}^h$, the individual is satiated if and only if he is satiated at the state $\overline{z}_h$. The individual is locally satiated if and only if he is locally satiated at $\overline{z}_h$.

Under the assumption of continuity, a quasi-equilibrium, (p^*, a^*), is a competitive equilibrium if there is no individual for whom z^{h*} is a minimum expenditure bundle at prices p^*. A competitive equilibrium, (p^*, a^*), is a quasi-equilibrium if no individual is locally satiated at z^{h*}.

A competitive equilibrium allocation or a quasi-equilibrium allocation is weakly pareto optimal, but not necessarily pareto optimal. A competitive equilibrium allocation which is also a quasi-equilibrium is pareto optimal.

A feasible allocation, a^*, which is weakly pareto optimal, need not be pareto optimal even if no individual is locally satiated at z^{h*}.

Assumption [Convexity]: The set of feasible social states is convex. For every individual, the domain of social states is convex and the utility function is quasi-concave.

If the assumption of convexity fails, competitive equilibria or quasi-equilibria need not exist.

Assumption [Full dimensionality]: The set of feasible social states has non-empty interior.

This is a regularity assumption. Under the assumption of convexity, it is without loss of generality: it suffices to consider the lowest dimensional affine space which contains the set of feasible social states and then rely on convexity to guarantee that the set has non-empty interior.

5. Under standard assumptions, the allocation associated with a social state which is weakly pareto optimal is a quasi-equilibrium allocation.

Lemma 2: Under the assumptions of compatibility, continuity, convexity and full dimensionality, the allocation associated with a social state which is weakly pareto optimal and at which no individual is locally satiated is a quasi-equilibrium allocation.

Proof:

The social state s^* is weakly pareto optimal and a^* is the associated allocation. No individual is locally satiated at s^* and, hence, at z^{h*}.

$\mathbf{D}(a^*) = \{x \in \mathbf{R}^{HN} : x = \sum_{h \in \mathbf{H}} z^h - y$, such that $y \in \mathbf{Y}$, while $z^h \in \mathbf{Z}^h$ and $u^h(z^h) > u^h(z^{h*})$, for $h \in \mathbf{H}\}$.

By the convexity assumption, the set $\mathbf{D}(a^*)$ is convex.

Since no individual is satiated at $a^*, \mathbf{D}(a^*) \neq \emptyset$.

Since the state s^* is weakly pareto optimal, $0 \notin \mathbf{D}(a^*)$.

It follows that there exists $p^* \neq 0$ such that $p^* x \geq 0$, for all $x \in \mathbf{D}(a^*)$.

If, for some individual, $u^{\hat{h}}(z^{\hat{h}}) \geq u^{\hat{h}}(z^{\hat{h}*})$, since individuals are not locally satiated at z^{h*}, there exist sequences $(z_n^h : n = 1, \cdots)$, for $h \in \mathbf{H}$, such that $\lim_{n \to \infty} z_n^h = z^{h*}$, for $h \in \mathbf{H} \setminus \{\hat{h}\}$, while $\lim_{n \to \infty} z_n^{\hat{h}} = z^{\hat{h}}$, and $x_n = \sum_{h \in \mathbf{H}} z_n^h - y^* \in \mathbf{D}(a^*)$, for $n = 1, \cdots$. It follows that $p^*(\sum_{h \in \mathbf{H}} z_n^h - y^*) \geq p^*(\sum_{h \in \mathbf{H}} z_n^{h*} - y^*) = 0$ for $n = 1, \cdots$, and hence $p^* z^{\hat{h}} \geq p^* z^{\hat{h}*}$

By a similar argument, $p^* y^* \geq p^* y$, for all $y \in \mathbf{Y}^*$.

It remains to show that, for some individual, $u^h(z^h) > u^h(z^{h*}) \Rightarrow p^* z^h > p^* z^{h*}$ or that z^{h*} is not a minimum wealth bundle for every individual. Arguing by contradiction, suppose $p^* z^h \geq p^* z^{h*}$, for all $z^h \in \mathbf{Z}^h$ and all $h \in \mathbf{H}$. By the compatibility assumption and profit maximization, it follows that $p^* y \geq p^* y^*$ for all $y \in \mathbf{Y}$. First, suppose $\sum_{h \in \mathbf{H}} p_h^* = 0$. Since $p^* \neq 0$, suppose $p_1^* \neq 0$. It follows that $p_1^* s \geq p_1^* s^*$ and $(\sum_{h \in \mathbf{H} \setminus \{1\}} p_h^*) s \geq (\sum_{h \in \mathbf{H} \setminus \{1\}} p_h^*) s^*$, for all $s \in \mathbf{S}$. Since $\sum_{h \in \mathbf{H}} p_h^* = 0, p_1^* s = p_1^* s^*$, for all $s \in \mathbf{S}$, which contradicts full dimensionality. If $\sum_{h \in \mathbf{H}} p_h^* \neq 0$, a contradiction follows by a similar argument.

∎

The allocation associated with a social state which is weakly pareto optimal, and at which some individual is locally satiated, need not be a quasi-equilibrium allocation.

Example 2: The set of feasible social states is $\mathbf{S} = \{s : 0 \leq s \leq 3\} \subset \mathbf{R}$. Individuals are $h \in \mathbf{H} = \{1,2\}$ and have utility functions $u^1 = \min\{s,1\}, u^2 = s$, with domain the set of feasible social states. The social state $s^* = 2$ is weakly pareto optimal, but not pareto optimal: it is dominated by the states in $\mathbf{S}^* = \{s \in \mathbf{S} : 2 < s \leq 3\}$. If $p^* = (p_1^*, p_2^*)$ is such that $p_1^* s \geq p_1^* s^*$ whenever $u^2(s) \geq u^2(s^*), p_1^* \geq 0$. It follows that $p_2^* \leq 0$. If $\Sigma_{h \in \mathbf{H}} p_h^* = 0$, as profit maximization requires since s^* is an interior state, and $p^* \neq 0, p_2^* < 0$. But then, for $s \in \mathbf{S}^1 = \{s \in \mathbf{S} : 2 < s \leq 3\}, u^1(s) = u^1(s^*)$, while $p_2^* s < p_2^* s^*$. ∎

Under the assumption of convexity, the allocation associated with a social state, not necessarily a pareto optimal one, at which some individual is satiated is a competitive equilibrium allocation.

Example 3: The set of feasible social states is $\mathbf{S} = \{s = (s_1, s_2) : 0 \leq s_1 \leq 2, 0 \leq s_2 \leq 2\} \subset \mathbf{R}^2$. Individuals are $h \in \mathbf{H} = \{1,2,3\}$ and have utility functions $u^1 = s_1 + s_2, u^2 = s_2 - s_1, u^3 = \min\{0, 1 - s_2\}$, with domain the set of feasible social states. The social state $s^* = (1,0)$ is not pareto optimal – it is dominated by states in $\mathbf{S}^* = \{s \in \mathbf{S} : s_1 + s_2 > 1, s_2 - s_1 > -1, s_2 \leq 1\}$ which is non-empty: $(1,1) \in \mathbf{S}^*$. Nevertheless, the satiation of individual $h = 3$ at s^* allows the associated allocation to obtain as a competitive equilibrium allocation at prices $p^* = (p_1^*, p_2^*, p_3^*)$, where $p_1^* = (1,1), p_2^* = (-1,1), p_3^* = (0,-2)$. ∎

Under the assumptions of continuity and convexity, a social state at which no individual is satiated yields the same utility to all individuals as another state at which at least one individual is not locally satiated.

Corollary 1: Under the assumptions of compatibility continuity, convexity and full dimensionality, the allocation associated with a social state which is pareto optimal, and at which no individual is satiated and at least one individual is not locally satiated, is a quasi-equilibrium allocation.

Proof: The social state s^* is weakly pareto optimal and a^* is the associated allocation. Individual $h = 1$ is not locally satiated at s^* and, hence, at z^{1*}.

The argument proceeds as in the proof of lemma 2, only with $\mathbf{D}(a^*) = \{x \in \mathbf{R}^{HN} : x = \sum_{h \in \mathbf{H}} z^h - y$, such that $y \in \mathbf{Y}$, while $z^h \in \mathbf{Z}^h$ and $u^h(z^h) \geq u^h(z^{h*})$, for $h \in \mathbf{H}$, with $u^1(z^1) > u^1(z^{1*})\}$. It suffices to observe that, since no individual is satiated at s^*, the set $\mathbf{D}(a^*)$ is non-empty, while $0 \notin \mathbf{D}(a^*)$. ∎

6. The allocation associated with a social state which is pareto optimal, even in the interior of the set of feasible social states, and which, according to lemma 2, is a quasi-equilibrium allocation, need not be a competitive equilibrium allocation.

Example 4: The set of feasible social states is $\mathbf{S} = \{s = (s_1, s_2) : 0 \leq s_1 \leq 2, -2 \leq s_2 \leq 2\} \subset \mathbf{R}^2$. Individuals are $h \in \mathbf{H} = \{1, 2, 3\}$ and have utility functions $u^1 = -\|s\|^2, u^2 = -\|s-(2,0)\|^2, u^3 = s_2$, with domain the set of feasible social states. The social state $s^* = (1, 0)$ is a pareto optimal social state where no individual is locally satiated. The associated allocation is a quasi-equilibrium allocation but not an equilibrium allocation. At $s^*, Du^1 = (-2, 0), Du^2 = (2, 0)$ and $Du^3 = (0, 1)$. Suppose $p^* = (p_1^*, p_2^*, p_3^*)$, with $\Sigma_{h=1}^3 p_h^* = 0$, as profit maximization requires are competitive equilibrium prices. For individual $h = 1, Du^1 ds > 0 \Rightarrow p_{1,1}^* ds_1 + p_{1,2}^* ds_2 > 0$ or $p_{1,1}^* < 0$ and $p_{1,2}^* = 0$. Similarly, for individual $h = 2, p_{2,1}^* > 0$ and $p_{2,2}^* = 0$. For individual $h = 3, Du^3 ds > 0 \Rightarrow p_{3,1}^* ds_1 + p_{3,2}^* ds_2 > 0$ or $p_{3,1}^* = 0$ and $p_{3,2}^* > 0$. But this leads to a contradiction, since $\Sigma_{h=1}^3 p_{h,2}^* > 0$. Nevertheless, the allocation is a quasi-equilibrium allocation at prices $p^* = (p_1^*, p_2^*, p_3^*)$, where $p_1^* = (-1, 0), p_2^* = (1, 0), p_3^* = (0, 0)$. ∎

A feasible social state, s^* irreducible if and only if, for any non-empty proper subset of individuals, $\hat{\mathbf{H}} \subset \mathbf{H}, \hat{\mathbf{H}} \neq, \mathbf{H}$, there exists a feasible social state $\hat{s}$ with $u^{\hat{\mathbf{H}}}(\hat{s}) > u^{\hat{\mathbf{H}}}(s^*)$.

The state s^* in example 5 (see figure 1) is not irreducible, since there is no state $\hat{s}$ such that $u^{\hat{\mathbf{H}}}(\hat{s}) > u^{\hat{\mathbf{H}}}(s^*)$, for $\hat{\mathbf{H}} = \{1, 2\}$.

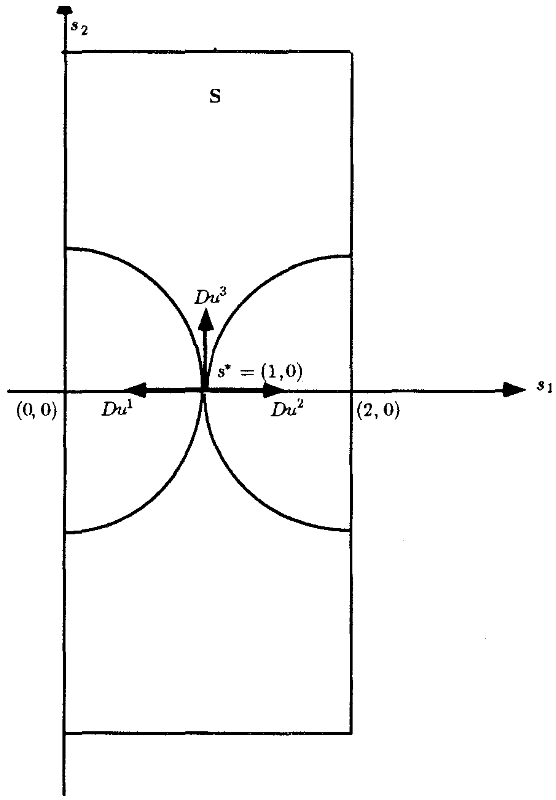

Figure 1

Proposition 1: Under the assumption of compatibility, convexity and full dimensionality, the allocation associated with a social state which is weakly pareto optimal and irreducible, and at which no individual is locally satiated, is a competitive equilibrium allocation.

Proof: The social state s^* is weakly pareto optimal and irreducible and a^* is the associated social state. No individual is locally satiated at s^* and, hence, at z^{h*}.

By lemma 2, there exists p^* such that (p^*, a^*) is a quasi-equilibrium, and $p^* \neq 0$.

In order to show that (p^*, a^*) is a competitive equilibrium it suffices to show that z^{h*} is not a minimum expenditure bundle for any individual at p^*.

The set of individuals for whom z^{h*} is a minimum expenditure bundle at p^* is $\hat{\mathbf{H}} = \{h \in \mathbf{H} : p^* z \geq p^* z^{h*}, \text{ for all } z \in \mathbf{Z}^h\} \subset \mathbf{H}$

By the concluding argument in the proof of lemma 2, $\hat{\mathbf{H}} \neq \mathbf{H}$. Arguing by contradiction : $\hat{\mathbf{H}} \neq \emptyset$. Since the state s^* is irreducible, and since $\mathbf{H} \backslash \hat{\mathbf{H}}$ is a non-empty, proper subset of individuals, there exists a state $\hat{s}$ such that $u^{\mathbf{H} \backslash \hat{\mathbf{H}}}(\hat{s}) > u^{\mathbf{H} \backslash \hat{\mathbf{H}}}(s^*)$. Since individuals in $\mathbf{H} \setminus \hat{\mathbf{H}}$ are not at a minimum expenditure bundle at (p^*, a^*), $\sum_{h \in \mathbf{H} \backslash \hat{\mathbf{H}}} p_h^* \hat{s} > \sum_{h \in \mathbf{H} \backslash \hat{\mathbf{H}}} p_h^* s^*$. Since individuals in $\hat{\mathbf{H}}$ are at a minimum expenditure at $(p^*, a^*), \sum_{h \in \hat{\mathbf{H}}} p_h^* \hat{s} \geq \sum_{h \in \hat{\mathbf{H}}} p_h^* s^*$. It follows that $\sum_{h \in \mathbf{H}} p_h^* \hat{s} > \sum_{h \in \mathbf{H}} p_h^* s^*$ or $p^* \hat{y} > p^* y^*$, for $\hat{y} = (\cdots, \hat{s}, \cdots)$, which contradicts profit maximization at p^*. ∎

As with quasi-equilibria, corollary 1, the requirement that no individual be locally satiated for the allocation associated with an irreducible social state to be a competitive equilibrium allocation can be relaxed to the requirement that at least one individual be locally non-satiated if the state is pareto optimal.

7. Applications follow.

First, an exchange economy with no external effects across individuals : Individuals are $h \in \mathbf{H}$ and commodity bundles are $x \in \mathbf{R}^L$, euclidean space of finite dimension. An individual is characterized by the exchange set $\mathbf{Z}^h \subset \mathbf{R}^L$, a set of z^h, exchange bundles, and by $u^h : \mathbf{Z}^h \rightarrow \mathbf{R}$, his ordinal utility function. The economy is

$$\mathbf{E} \;=\; \{\mathbf{H}, (\mathbf{Z}^h, u^h) : h \in \mathbf{H}\}.$$

An allocation is an array $\alpha = (\cdots, z^h, \cdots)$ of consumption bundles, $z^h \in \mathbf{Z}^h$, for each individual. An allocation is feasible if and only if $\sum_{h \in \mathbf{H}} z^h = 0$.

The society associated with the economy $\mathbf{E}$ is

$$\boldsymbol{\Sigma}(\mathbf{E}) \;=\; \{\mathbf{H}, (\mathbf{S}^h, u^h) : h \in \mathbf{H}, \mathbf{S}\},$$

where a social state is $s = (\cdots, z_h, \cdots) \in \mathbf{R}^N$ with $N = HL$, the domain of social states for an individual is $\mathbf{S}^h = \{s = (\cdots, z_{h'}, \cdots, z_h, \cdots) : z_h \in \mathbf{Z}^h\}$, the utility function of individual is $u^h(s) = u^h(z_h)$, and the set of feasible social states is $\mathbf{S} = \{s = (\cdots, z_h, \cdots) : z_h \in \mathbf{Z}^h, \text{ for } h \in \mathbf{H}, \text{ and } \sum_{h \in \mathbf{H}} z_h = 0\}$.

The economy associated with the society $\boldsymbol{\Sigma}(\mathbf{E})$ is $\mathbf{E}(\boldsymbol{\Sigma}(\mathbf{E}))$ with exchange bundles $\tilde{z} = (\cdots, \tilde{z}_h, \cdots) = (\cdots, z_{h,h'}, \cdots)$ and prices $p = (\cdots, p_h, \cdots) = (\cdots, p_{h,h'}, \cdots)$. From the structure of the exchange sets of individuals and of the technology, it follows that at a competitive equilibrium, $p_{h,h'} = 0$, for $h \neq h'$, while $p_{h,h,} = p_{h',h'}$.

In this set up, the condition of irreducibility coincides with the condition studied in McKenzie (1959).

Thus, competitive equilibria for the economy associated with a society which is in turn associated with an underlying economy coincide with the competitive equilibria for the underlying economy, which demonstrates the consistency of the construction.

Second, an exchange economy with external effects across individuals : individuals are $h \in \mathbf{H}$ and commodity bundles are $x \in \mathbf{R}^L$, the euclidean space of finite dimension. An individual is characterized by the exchange set $\mathbf{Z}^h \subset \mathbf{R}^L$, a set of z^h, exchange bundles, and by $u^h : \mathbf{X}_{h \in \mathbf{H}} \mathbf{Z}^h \to \mathbf{R}$, his ordinal utility function. The economy is

$$\mathbf{E} \; = \; \{\mathbf{H}, (\mathbf{Z}^h, u^h) : h \in \mathbf{H}\}.$$

An allocation is an array $\alpha = (\cdots, z^h, \cdots)$ of consumption bundles, $z^h \in \mathbf{Z}^h$, for each individual. An allocation is feasible if and only if $\sum_{h \in \mathbf{H}} z^h = 0$.

The society associated with this economy is

$$\mathbf{\Sigma}(\mathbf{E}) \; = \; \{\mathbf{H}, (\mathbf{S}^h, u^h) : h \in \mathbf{H}, \mathbf{S}\}$$

where a social state is $s = (\cdots, z^h, \cdots) \in \mathbf{R}^N$, $N = HL$, the domain of social states for an individual is $\mathbf{S}^h = \{s = (\cdots, z_h, z_{h'}, \cdots) : z_h \in \mathbf{Z}^h\}$ and utility-function is $u^h(s) = u^h(\cdots, z_h, z_{h'}, \cdots)$ and the set of feasible social states is

$$\mathbf{S} \; = \; \{s = (\cdots, z_h, \cdots) : z_h \in \mathbf{Z}^h, h \in \mathbf{H}, \sum_{h \in \mathbf{H}} z_h = 0\}.$$

The economy associated with the society $\mathbf{\Sigma}(\mathbf{E})$ is $\mathbf{E}(\mathbf{\Sigma}(\mathbf{E}))$, with exchange bundles $\tilde{z} = (\cdots, \tilde{z}_h, \cdots) = (\cdots, z_{h,h'}, \cdots)$ and prices $p = (\cdots, p_h, \cdots) = (\cdots, p_{h,h'}, \cdots)$. When, the consumption externalities are negative, the markets in $\mathbf{E}(\mathbf{\Sigma}(\mathbf{E}))$ correspond to "markets for pollution rights". In general, the markets in $\mathbf{E}(\mathbf{\Sigma}(\mathbf{E}))$ correspond to "markets for externalities".

Opening markets for externalities may not decentralize a pareto optimal allocation if the associated social state fails to be a irreducible.

Example 6: There is a single commodity. Individuals are $h \in \mathbf{H} = \{1, 2, 3\}$ and have utility functions, $u^1 = z_1 + 1, u^2 = (z_2 + \frac{1}{2}) - (z_3 + \frac{1}{2})$, and $u^3 = (z_3 + \frac{1}{2}) - (z_2 + \frac{1}{2})$. For all individuals, $S^h = \{(z_1, z_2, z_3) \in \mathbf{R}^3 : z_1 \geq -1, z_2 \geq -\frac{1}{2}, z_3 \geq -\frac{1}{2}\}$, while the set of feasible social states is $\mathbf{S} = \{(z_1, z_2, z_3) : z_1 + z_2 + z_3 = 0, z_1 \geq -1, z_2 \geq -\frac{1}{2}, z_3 \geq -\frac{1}{2}\} \subset \mathbf{R}^3$. The social state $s^* = (0, 0, 0)$ is Pareto optimal, but it is not irreducible which can be seen by setting $\mathbf{H}^1 = \{1\}$ and $\mathbf{H}^2 = \{2, 3\}$. Indeed, s^* can be supported as a quasi-equilibrium but not as a competitive equilibrium. As s^* is in the interior of $\mathbf{S}$, to support s^* as a competitive equilibrium, from the profit maximization of the firm, $\sum_{h \in \mathbf{H}} p_h^* = 0$. At s^*, $Du^1 = (1, 0, 0), Du^2 = (0, 1, -1)$ and $Du^3 = (0, -1, 1)$. For individual $h = 1, Du^1 ds > 0 \Rightarrow p_{1,1}^* ds_1 + p_{1,2}^* ds_2 + p_{1,3}^* ds_3 > 0$, requires that $p_{1,1}^* > 0, p_{1,2}^* = p_{1,3}^* = 0$. For individual $h = 2, Du^2 ds > 0 \Rightarrow p_{2,1}^* ds_1 + p_{2,2}^* ds_2 + p_{2,3}^* ds_3 > 0$ requires that $p_{2,1}^* = 0, p_{2,2}^* > 0$ and $p_{2,3}^* < 0$. For individual $h = 3$, $Du^3 ds > 0 \Rightarrow p_{3,1}^* ds_1 + p_{2,2}^* ds_2 + p_{2,3}^* ds_3 > 0$ requires that $p_{3,1}^* = 0, p_{3,2}^* < 0, p_{3,3}^* > 0$. But this leads to a contradiction as $\sum_{h \in \mathbf{H}} p_{h,1}^* > 0$. Nevertheless, s^* can be supported as a quasi-equilibrium at prices $p^* = (p_1^*, p_2^*, p_3^*) = (0, 0, 0, 0, 1, -1, 0, -1, 1)$.

Third, a game in normal form : the "prisoners' dilemma" with players $h \in \mathbf{H} = \{1, 2\}$, strategies $\{C^h, NC^h\}$, to cooperate or not cooperate, for each player and payoffs

	C^1	NC^1
C^2	$(2, 2)$	$(-1, 3)$
NC^2	$(3, -1)$	$(1, 1)$

Associated with this game there is a society with the players as individuals, the probability distributions over payoffs, $s = (s_1, s_2, s_3)$, with $s_1, s_2, s_3 \geq 0, s_1 + s_2 + s_3 \leq 1$, according to

	C^1	NC^1
C^2	s_1	s_2
NC^2	s_3	$1-(s_1+s_2+s_3)$

as social states, and the expected payoffs of players as the utility functions of individuals,

$$\begin{aligned} u^1 &= 2s_1 - s_2 + 3s_3 + 1 - (s_1 + s_2 + s_3) = s_1 + 2(s_3 - s_2) + 1, \\ u^2 &= 2s_1 + 3s_2 - s_3 + 1 - (s_1 + s_2 + s_3) = s_1 + 2(s_2 - s_3) + 1. \end{aligned}$$

A Nash equilibrium[17] social state is $\overline{s} = (0, 0, 0)$, which is not pareto optimal.

Probability distributions over payoffs define social states instead of mixed strategy tuples. In the economy associated with the set of social states, each player chooses a social state. Therefore, even though a player's utility is linear in his own choice of mixed strategies (given the choice of mixed strategy by other players), it is not quasi-concave in the mixed-strategy profile. Using mixed strategy tuples to define social states implies that utility functions fail to quasi-concave in the economy associated with the set of social states. Probability distributions over payoffs can be justified if we assume, as in the analysis of correlated equilibria, that players correlate their choice of strategies by using a correlation device whose recommendations we based on publicly observed signals.

In this game, the only correlated equilibrium based on public signals is the social state $\bar{s} = (0, 0, 0)$.

Pareto optimal social states are characterized by $s_1^* + s_2^* + s_3^* = 1$, and $s_2^* s_3^* = 0$. Each can be supported as a competitive equilibrium with competitive equilibrium prices $p^* = (p_1^*, p_2^*) = (0, 2, -2, 0, -2, 2)$.

By trading in probabilities or strategies, individuals can attain pareto optimality.

[17] Nash (1949).

References

Arrow, K.J. (1970), "Political and economic evaluation of social effects and externalities," in M. Intriligator (ed.), *Frontiers of Quantitative Economics*, North Holland Publishing Company, 3-25.

Arrow, K.J. (1951), "An extension of the basic theorems of classical welfare economics," in J. Neyman (ed.), *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 507-532.

Arrow, K.J. and G. Debreu (1954), "Existence of equilibrium for a competitive economy," *Econometrica*, 22, 265-290.

Coase, R.H. (1960), "The problem of social cost," *Journal of Law and Economics*, 3, 1-44.

Debreu, G. (1951), "The coefficient of resource utilization," *Econometrica*, 19, 273-292.

Foley, D. (1970), "Lindahl's solution and the core of an economy," *Econometrica*, 38, 66-72.

Groves, T. and J. Ledyard (1977), "Optimal allocation of public goods: a solution to the 'free rider' problem," *Econometrica*, 45, 783-000.

Lindahl, E. (1919), "Positive Lösung," in E. Lindahl, *Die Gerechtigkeit der Besteuerung*, xxxx, 85-89.

Lindahl, E. (1928), "Einige strittige Fragen der Steuertheorie," in H. Meyer (ed.), *Die Wirtschaftstheorie der Gegenwart*, xxxx, vol.IV, 282-304.

McKenzie, L. (1959), "On the existence of general equilibrium for a competitive economy," *Econometrica*, 27, 54-71.

McKenzie, L. (1961), "On the existence of general equilibrium: some corrections," *Econometrica*, 29, 247-248.

Mas-Colell, A. (1980), "Efficiency and decentralization in the pure theory of public goods," *Quarterly Journal of Economics*, 94, 625-641.

Milleron, J.-C. (1972), "Theory of value with public goods: a survey article," *Journal of Economic Theory*, 5, 419-477.

Nash, J.F. (1949), "Equilibrium points in n-person games," *Proceedings of the National Academy of Sciences* (U.S.A.), 36-48.

Starrett, D. and P. Zeckhauser (1974), "Treating external diseconomies-markets or taxes," in J.W. Pratt (ed.) *Statistical and Mathematical Aspects of Pollution Problems*, Dekker, 65-84.